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## **Observer-based adaptive neural tracking control for a class of stochastic nonlinear systems**

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## **Observer-based adaptive neural tracking control for a class of stochastic nonlinear systems**

In this paper, an observer-based adaptive neural tracking control approach is proposed for a class of stochastic nonlinear systems with immeasurable states. The radial basis function neural networks (RBFNNs) are used to approximate the unknown nonlinear functions, and a linear reduced-order state observer is

designed for estimating the unmeasured states. Based on the designed the state observer, an adaptive neural output feedback control approach is developed via backstepping control design. It is shown that the proposed controller ensures that all signals of the closed-loop system remain bounded in probability, and the tracking error converges to an arbitrarily small neighborhood around the origin in the sense of mean quartic value. Finally, two examples are given to illustrate the effectiveness of the proposed design approach.

Keywords: stochastic nonlinear systems; neural networks; reduced-order state observer; adaptive tracking control; output feedback

## 1 Introduction

Over recent years, intensive research activities have been reported for the controller design and stability analysis of stochastic nonlinear systems (Guo, Zuo, Liu, & Liang, 2014; Ji & Xi, 2006; Liu & Zhang, 2008). Many results obtained for deterministic systems have been successfully extended to the stochastic setting, such as Lyapunov stability theory (Florchinger, 1995), LaSalle invariance principle (Mao, 1999), backstepping technique (Deng & Krstic, 1997; Pan & Basar, 1999; Sun et al., 2016) etc. Among them, backstepping approach has become an important control approach to control various systems and processes. Pan and Basar (1998) first generalized backstepping approach to the stochastic nonlinear systems in order to deal with the problem of asymptotically tracking for a class of stochastic systems under a risk-sensitive cost. Since then, a series of extensions have been made for different stochastic nonlinear systems, for example, Liu (2014) proposed an observer-based control approach for a class of stochastic nonlinear systems with strict-feedback form. Li, Jing, and Zhang (2011) dealt with the stabilization problem for a class of

high-order stochastic nonlinear systems. Based on the simultaneous domination approach, Wu, Chen, Li, and Ren (2013) investigated the stabilization problem for a class of switched stochastic nonlinear systems. Gao, Sun, and Wang (2017) applied decentralized state observers and backstepping technique dealt with the problem of decentralized adaptive tracking control of a class of large-scale stochastic nonlinear systems. Xie and Xie (2000) designed a robust decentralized controller for a class of large-scale interconnected bilinear stochastic systems with time delays. Song and Zhai (2018) proposed an observer-based output-feedback control strategy for a class of switched stochastic nonlinear systems with time-delay and uncertain output function. Correspondingly, the analysis and synthesis problems of stochastic nonlinear systems, especially tracking and stability analysis, have been widely investigated (Ji & Xi, 2006; Psillakis & Alexandridis, 2006) and the references therein. Compared with the general nonlinear systems, the investigation on stability analysis and controller design for stochastic nonlinear systems is a rather challenging problem. What's the worse, many stochastic nonlinear systems fail to preserve tracking performance and stability due to the fact that the nonlinear systems are disturbed by stochastic disturbance. Therefore, it is of important theoretical and practical significance to study the stochastic nonlinear systems.

In recent years, approximation-based technique has become a popular design approach for the control of nonlinear systems, such as multidimensional Taylor network (MTN) (Han, 2018a; Han & Yan, 2018a, 2018b; Yan, Han, & Sun, 2018), neural network (NN) (Gao, Liu, Liu, & Li, 2018; Wang, Chen, & Lin, 2012, 2014),

biological NNs in (Wang, Liu, Zhang, & Zhao, 2016), etc. Among them, NN-based adaptive control schemes have become the significant design approaches in the control community, and many interesting results have been reported, for example, Miao and Li (2015) proposed a robust adaptive tracking control approach for a class of single-input single-output (SISO) nonlinear systems with strict-feedback. Li and Li (2017) presented an adaptive neural tracking control strategy for a class of uncertain SISO nonlinear systems with the full state constraints and time-delays. Han (2018b) proposed an adaptive neural tracking control approach for a class of SISO nonlinear systems with dynamic uncertainties. Liu and Tang (2018) proposed a barrier Lyapunov function (BLF)-based controller to solve the partial state constraint problems for a class of SISO nonlinear systems with backlash-like hysteresis. Liu, Liu, and Tong (2019) studied the finite time fault tolerant control problem for a class of switched nonlinear systems. Chen, Ge, and How (2010), Chen, Li, and Chen (2017) and Tong, Li, and Zhang (2011) addressed the control problems for several classes of multi-input and multi-output (MIMO) nonlinear systems under different assumptions. Khoo, Yin, Man, and Yu (2013) and Yin, Khoo, Man, and Yu (2011) discussed the finite-time stability and instability problems for stochastic nonlinear systems. Wang et al. (2014) proposed a NN-based control approach for a class of stochastic nonlinear systems with backlash-like hysteresis nonlinearities. Li, Li, and Feng (2016) investigated the adaptive control problem for SISO stochastic nonlinear time-delay systems with unknown dead-zone. Xia and Zhang (2019) and Si, Dong, and Yang (2017) developed NN-based adaptive control strategies for two classes of full state

constrained stochastic nonlinear systems. Li et al. (2018) developed a novel NN-based adaptive control approach for a class of stochastic nonlinear systems with multiple time-varying delays and input dead-zone. Besides, many NN-based adaptive control approaches have been developed to MIMO stochastic nonlinear systems (Li, Chen, & Li, 2011; Ning, Yu, Xing, & Wang, 2016; Zhou, Shi, Liu, & Xu, 2012). Among them, the problem of adaptive output feedback control for stochastic nonlinear systems has attracted increasing attention in recent years. Chen et al. (2010) addressed the problem of adaptive output-feedback control for a class of stochastic nonlinear systems with time-varying delays. Tong, Wang, Li, and Zhang (2014) discussed the problem of adaptive neural network output feedback control for a class of stochastic nonlinear strict-feedback systems with both unmodeled dynamics and unknown dead zone. Yu and Li (2014) discussed the problem of output-feedback adaptive stabilization for a class of stochastic nonlinear time-delay systems with unknown control directions. Wang et al. (2015) considered the problem of observer-based adaptive neural output-feedback control for a class of stochastic nonlinear systems with nonstrict-feedback structure. For a class of stochastic nonlinear systems with time-varying delay and unknown control directions, Li, Li, Wang, and Chen (2015), presents an NN-based adaptive output-feedback control scheme. Li and Liu (2015) considered the global stabilization via time-varying output-feedback for a class of stochastic nonlinear systems possessing uncertain control coefficients and unmeasured states dependent with unknown growth rate. Based BLF method, Xu, Liang, Li, and Ma (2019) proposed an adaptive output-feedback controller for

nonlinear strict-feedback stochastic systems with input saturation and output constraint. Yu, Li, Yu, and Li (2018) investigated the problem of output feedback adaptive stabilization for a class of nonstrict-feedback stochastic nonlinear systems with unknown backlash-like hysteresis and unknown control directions. Therefore, the investigation on stability analysis and control design of stochastic nonlinear systems is crucial for control theory, as well as the synthesis of practical control systems. In fact, it is still a meaningful and challenging task to further study on the adaptive output feedback controller design problem of stochastic nonlinear systems is of great significance, which motivates our present research.

Motivated by the above observation, this paper focuses on the problem of adaptive neural tracking control for stochastic nonlinear systems with immeasurable states. First, a linear reduced-order observer is first introduced to estimate the unmeasured states. Second, combining the backstepping technique and adaptive control design, a novel observer-based adaptive neural backstepping control approach is developed. It is proved that all the signals of the closed-loop system are bounded in probability and the tracking errors finally converge to a small neighborhood of the origin. The main contributions of this paper are listed as follows:

i) This paper investigated the tracking control problem for a class of stochastic nonlinear systems. The proposed control scheme cannot only guarantee the stability of the whole stochastic nonlinear, but also can lead to a satisfactory tracking control performance.

ii) A novel backstepping-based adaptive neural output-feedback controller is

proposed for a class of stochastic nonlinear systems. Moreover, by designing a linear reduced-order observer, the proposed adaptive control method does not require that all the states of the system are measured directly.

iii) We lump all unknown functions into a suitable unknown function that is estimated by only an RBFNN in each step of the backstepping, and therefore, the computation burden is greatly reduced.

This paper is organized as follows. Section 2 provides some preliminaries and problem formulation. Section 3 presents the design of reduced-order observer. A novel NN-based adaptive output-feedback control scheme is presented in Section 4. Section 5 gives two simulation examples to illustrate the effectiveness of our results. In Section 6, we conclude the work of this paper.

## 2 Preliminaries and problem formulation

The following notations are used throughout this paper.  $\mathbb{R}_+$  denotes the set of all non-negative real numbers,  $\mathbb{R}^n$  indicates the real  $n$ -dimensional space,  $\mathbb{R}^{n \times r}$  denotes the set of all  $n \times r$  real matrices. For a given vector or matrix  $X$ ,  $X^T$  stands for its transpose and  $\text{Tr}(X)$  denotes its trace when  $X$  is square and  $\|X\|$  denotes the Euclidean norm of a vector  $X$ ,  $\mathcal{C}^i$  denotes the set of all functions with continuous  $i$ -th partial derivative.  $I_i$  is  $i \times i$  identity matrix.  $P\{\mathcal{A}\}$  represents the probability of event  $\mathcal{A}$ .

### 2.1 Problem descriptions

Consider the following stochastic nonlinear system

$$\begin{cases} dx_i = (x_{i+1} + f_i(y))dt + g_i(y)^T d\omega \\ 1 \leq i \leq n-1 \\ dx_n = (u + f_n(y))dt + g_n(y)^T d\omega \\ y = x_1 \end{cases} \quad (1)$$

where  $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$ ,  $u \in \mathbb{R}$  and  $y \in \mathbb{R}$  are the system state vector, the control input and the system output, respectively.  $\omega$  denotes an  $r$ -dimensional standard Brownian motion defined on the complete probability space  $(\mathcal{W}, \mathcal{F}, \mathcal{P})$  with  $\mathcal{W}$  being a sample space,  $\mathcal{F}$  being a  $\sigma$ -field, and  $\mathcal{P}$  being a probability measure.  $f_i(\cdot): \mathbb{R} \rightarrow \mathbb{R}$  and  $g_i(\cdot): \mathbb{R} \rightarrow \mathbb{R}^r$  ( $i=1, \dots, n$ ) are unknown smooth functions with  $f_i(0) = 0$ ,  $g_i(0) = 0$ .

**Remark 2.1:**  $f_i(0) = 0$ ,  $g_i(0) = 0$  imply that the origin is the equilibrium of the system (2.1). According to the well-known *Mean Value Theorem*, the following equalities hold

$$f_i(y) = y\phi_i(y) \quad (2)$$

$$g_i(y) = y\varphi_i(y) \quad (3)$$

where  $\phi_i(y)$  and  $\varphi_i(y)$  are completely unknown nonlinear functions.

The control objective is to design an adaptive neural controller such that all the signals in the closed-loop system remain bounded, and the system output  $y$  follows the given reference signal  $y_d$ .

**Assumption 2.1:** The reference signal  $y_d$  and its time derivatives up to the  $n$ -th order are continuous and bounded.

**Assumption 2.2:** It is assumed that only output  $y$  is available for measurement.

## 2.2 Stochastic stability

Consider the following stochastic nonlinear system

$$dx = f(x)dt + g(x)d\omega \quad (4)$$

where  $x \in \mathbb{R}^n$  is the system state vector,  $\omega$  is a  $r$ -dimensional independent standard Wiener process,  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $g: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times r}$  are nonlinear continuous functions and satisfy  $f(0) = 0$ ,  $g(0) = 0$ .

**Definition 2.1** (Liu & Zhang, 2004): Consider the system (4), the equilibrium  $x = 0$  is said to be globally asymptotically stable in probability if for any  $t_0 \geq 0$  and  $\varepsilon > 0$ ,

$$\lim_{x(t_0) \rightarrow 0} P \left\{ \sup_{t \geq t_0} \|x(t)\| > \varepsilon \right\} = 0 \quad (5)$$

and for any initial condition  $x(t_0)$ ,

$$P \left\{ \lim_{t \rightarrow \infty} x(t) = 0 \right\} = 1 \quad (6)$$

**Definition 2.2** (Has'Minskii, 1980): The solution process  $\{x(t), t \geq 0\}$  of stochastic system (1) is said to be bounded in probability, if

$$\limsup_{\zeta \rightarrow 0} P \left\{ \|x(t)\| > \zeta \right\} = 0 \quad (7)$$

**Definition 2.3:** For any given  $V \in \mathcal{C}^2$ , associated with the stochastic differential equation (4), define the differential operator  $\mathcal{L}$  as follows

$$\mathcal{L}V = \frac{\partial V}{\partial x} f + \frac{1}{2} \text{Tr} \left( g^T \frac{\partial^2 V}{\partial x^2} g \right) \quad (8)$$

**Lemma 2.1** (Wang, Chen, & Lin, 2014): Consider the stochastic system (4). If there exists a positive definite, radially unbounded, twice continuously differentiable Lyapunov function  $V(x): \mathbb{R}^n \rightarrow \mathbb{R}$ , and constants  $a_0 > 0$  and  $b_0 \geq 0$ , such that

$$\mathcal{L}V(x) \leq -a_0V(x) + b_0 \quad (9)$$

then (i) the system has a unique solution almost surely, and (ii) the system is bounded in probability.

### 2.3 RBF neural network

In this paper, RBFNN will be used to approximate the unknown smooth nonlinear functions. Then, we have the following lemma.

**Lemma 2.2:** Assume that  $f(\mathbf{Z})$  is a continuous function over a compact set  $\Omega_{\mathbf{Z}} \in \mathbb{R}^q$ , then for  $\forall \varepsilon > 0$ , there exists a RBFNN  $\boldsymbol{\theta}^{*T} S(\mathbf{Z})$ , such that

$$f(\mathbf{Z}) = \boldsymbol{\theta}^{*T} S(\mathbf{Z}) + \delta(\mathbf{Z}), \quad |\delta(\mathbf{Z})| \leq \varepsilon \quad (10)$$

where  $\boldsymbol{\theta}^*$  is the ideal constant weight vector and defined by

$$\boldsymbol{\theta}^* = \arg \min_{\boldsymbol{\theta} \in \mathbb{R}^l} \left\{ \sup_{\mathbf{Z} \in \Omega_{\mathbf{Z}}} |f(\mathbf{Z}) - \boldsymbol{\theta}^T S(\mathbf{Z})| \right\} \quad (11)$$

where  $\delta(\mathbf{Z})$  denotes the approximation error.  $\boldsymbol{\theta} = [\theta_1, \dots, \theta_l]^T \in \mathbb{R}^l$  is the weight vector,  $S(\mathbf{Z}) = [s_1(\mathbf{Z}), \dots, s_l(\mathbf{Z})]^T$  is the basis function vector with  $l \in \mathbb{R}_+$  being the number of the NN nodes and  $s_i(\mathbf{Z})$  is chosen as the commonly used Gaussian function with the form as

$$s_i(\mathbf{Z}) = \exp\left[-(\mathbf{Z} - \boldsymbol{\mu}_i)^T (\mathbf{Z} - \boldsymbol{\mu}_i) / \gamma_i^2\right], \quad i = 1, \dots, l \quad (12)$$

with  $\boldsymbol{\mu}_i = [\mu_{i,1}, \dots, \mu_{i,q}]^T$  is the center of the receptive field, and  $\gamma_i$  is the width of the Gaussian function.

### 3 Design of reduced-order observer

Since the states  $x_2, \dots, x_n$  are not measured, first, we introduce the following linear reduced-order observer to estimate the unmeasured states (Liu, Zhang, & Jiang, 2007):

$$\begin{cases} \dot{\hat{x}}_i = \hat{x}_{i+1} + k_{i+1}y - k_i(\hat{x}_i + k_1y) \\ 1 \leq i \leq n-2 \\ \dot{\hat{x}}_{n-1} = u - k_{n-1}(\hat{x}_1 + k_1y) \end{cases} \quad (13)$$

where  $k_i$  are real numbers such that the following matrix is stable:

$$A = \begin{bmatrix} -k_1 & & & \\ -k_2 & & I_{(n-2) \times (n-2)} & \\ \vdots & & & \\ -k_{n-1} & 0 & \cdots & 0 \end{bmatrix}_{(n-1) \times (n-1)}$$

therefore, there exists a positive definite matrix  $P$  such that  $A^T P + PA = -I$ .

Define  $\tilde{\mathbf{x}} = [\tilde{x}_2, \dots, \tilde{x}_n]^T$  with the components  $\tilde{x}_i$  be given as

$$\tilde{x}_i = x_i - \hat{x}_{i-1} - k_{i-1}y, \quad i = 2, \dots, n \quad (14)$$

Let

$$F = \begin{pmatrix} f_2 - k_1 f_1 \\ f_3 - k_2 f_1 \\ \vdots \\ f_n - k_{n-1} f_1 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} g_2^T - k_1 g_1^T \\ g_3^T - k_2 g_2^T \\ \vdots \\ g_n^T - k_{n-1} g_n^T \end{pmatrix}$$

Then we get the following error dynamics:

$$d\tilde{\mathbf{x}} = (A\tilde{\mathbf{x}} + F)dt + Gd\omega \quad (15)$$

Consider the quartic Lyapunov function candidate

$$V_0 = \frac{1}{2}(\tilde{\mathbf{x}}^T P \tilde{\mathbf{x}})^2 \quad (16)$$

Then, by observer error dynamics system (15) and Definition 3, we have

$$\begin{aligned} \mathcal{L}V_0 &= (\tilde{\mathbf{x}}^T P \tilde{\mathbf{x}}) \left\{ \tilde{\mathbf{x}}^T (A^T P + PA) \tilde{\mathbf{x}} + 2\tilde{\mathbf{x}}^T P F \right\} \\ &\quad + 2\text{Tr} \left\{ G^T (2P\tilde{\mathbf{x}}\tilde{\mathbf{x}}^T P + \tilde{\mathbf{x}}^T P \tilde{\mathbf{x}} P) G \right\} \\ &= -(\tilde{\mathbf{x}}^T P \tilde{\mathbf{x}}) \tilde{\mathbf{x}}^T \tilde{\mathbf{x}} + 2\tilde{\mathbf{x}}^T P \tilde{\mathbf{x}} P F \\ &\quad + 2\text{Tr} \left\{ G^T (2P\tilde{\mathbf{x}}\tilde{\mathbf{x}}^T P + \tilde{\mathbf{x}}^T P \tilde{\mathbf{x}} P) G \right\} \end{aligned} \quad (17)$$

According (2) and (3), for each  $j = 2, \dots, n$ , we have

$$f_j - k_{i-1}f_1 = y\phi_j - k_{i-1}y\phi_1 = y(\phi_j - k_{i-1}\phi_1) = y\tilde{\phi}_j \quad (18)$$

$$g_j^T - k_{j-1}g_{j-1}^T = y\varphi_j^T - k_{j-1}y\varphi_{j-1}^T = y(\varphi_j^T - k_{j-1}\varphi_{j-1}^T) = y\tilde{\varphi}_j^T \quad (19)$$

where  $\tilde{\phi}_j = \phi_j - k_{i-1}\phi_1$  and  $\tilde{\varphi}_j^T = \varphi_j^T - k_{j-1}\varphi_{j-1}^T$ .

Then, combining Young's inequality with (18) and (19) gives

$$\begin{aligned} \tilde{x}^T P \tilde{x} \tilde{x}^T P F &\leq \|P\|^2 \|\tilde{x}\|^3 \|F\| \leq \sum_{j=2}^n \left( \|P\|^2 \|\tilde{x}\|^3 |f_j - k_{j-1}f_1| \right) \\ &= \sum_{j=2}^n \left( \|P\|^2 \|\tilde{x}\|^3 |y\tilde{\phi}_j| \right) \\ &\leq \sum_{j=2}^n \left( \frac{3}{4} \|P\|^{\frac{8}{3}} \|\tilde{x}\|^4 + \frac{1}{4} y^4 \tilde{\phi}_j^4 \right) \end{aligned} \quad (20)$$

$$\begin{aligned} \text{Tr} \{ G^T (2P\tilde{x}\tilde{x}^T P + \tilde{x}^T P \tilde{x} P) G \} &\leq 3 \|P\|^2 \|\tilde{x}\|^2 \|G\|_F^2 \\ &= 3 \sum_{j=2}^n \|P\|^2 \|\tilde{x}\|^2 \|g_j^T - k_{j-1}g_{j-1}^T\|^2 \\ &= 3 \sum_{j=2}^n \|P\|^2 \|\tilde{x}\|^2 y^2 \|\tilde{\varphi}_j\|^2 \\ &\leq 3 \sum_{j=2}^n \left( \frac{1}{2} \|P\|^4 \|\tilde{x}\|^4 + \frac{1}{2} y^4 \|\tilde{\varphi}_j\|^4 \right) \end{aligned} \quad (21)$$

Substituting (20) and (21) into (17) gives

$$\mathcal{L}V_0 \leq -\Pi \|\tilde{x}\|^4 + \sum_{j=2}^n \varpi_j y^4 \quad (22)$$

where  $\Pi = \lambda_{\min}(P) - \sum_{j=2}^n \left( \frac{3}{2} \|P\|^{\frac{8}{3}} + 3 \|P\|^4 \right)$  and  $\varpi_j = \frac{1}{2} \tilde{\phi}_j^4 + 3 \|\tilde{\varphi}_j\|^4$ .

#### 4 Adaptive NN output-feedback controller design

Based on stochastic nonlinear system (1) and reduced-order observer (13), one can express the entire system as

$$\begin{cases} d\tilde{\mathbf{x}} = (A\tilde{\mathbf{x}} + F)dt + Gd\omega \\ dy = (\hat{x}_1 + k_1 y + \tilde{x}_2 + f_1)dt + g_1^T d\omega \\ d\hat{x}_1 = (\hat{x}_2 + k_2 y - k_1(\hat{x}_1 + k_1 y))dt \\ d\hat{x}_i = (\hat{x}_{i+1} + k_{i+1} y - k_i(\hat{x}_1 + k_1 y))dt \\ \vdots \\ d\hat{x}_n = (u - k_n(\hat{x}_1 + k_1 y))dt \end{cases} \quad (23)$$

Then, the control strategy controller for system (23) can be designed by

Backstepping technique. Introduce a new state transformation

$$\begin{aligned} z_1 &= y - y_d \\ z_i &= \hat{x}_{i-1} - \alpha_{i-1} \end{aligned} \quad (24)$$

where  $i = 2, \dots, n$  and  $\alpha_{i-1}$  are the virtual signals to be designed.

#### 4.1 Adaptive NN-based controller design

*Step 1:* Consider a stochastic Lyapunov function candidate as

$$V_1 = V_0 + \frac{1}{4} z_1^4 + \frac{1}{2} \tilde{\boldsymbol{\theta}}_1^T \Gamma_1^{-1} \tilde{\boldsymbol{\theta}}_1 \quad (25)$$

where  $\tilde{\boldsymbol{\theta}}_1 = \boldsymbol{\theta}_1 - \hat{\boldsymbol{\theta}}_1$  is the parameter error and  $\Gamma_1 = \Gamma_1^T > 0$  is any constant matrix.

By Definition 3, we have

$$\mathcal{L}V_1 = \mathcal{L}V_0 + z_1^3 (\hat{x}_1 + k_1 y + \tilde{x}_2 + f_1 - \dot{y}_d) + \frac{3}{2} z_1^2 g_1^T g_1 - \tilde{\boldsymbol{\theta}}_1^T \Gamma_1^{-1} \dot{\tilde{\boldsymbol{\theta}}}_1 \quad (26)$$

Furthermore, according to Young's inequality, we have

$$\frac{3}{2} z_1^2 g_1^T g_1 \leq \frac{3}{4} \xi_1^2 + \frac{3}{4 \zeta_1^2} z_1^4 \|g_1\|^4 \quad (27)$$

$$z_1^3 \tilde{x}_2 \leq \frac{3}{4} \zeta_1^{\frac{4}{3}} z_1^4 + \frac{1}{4 \zeta_1^4} \tilde{x}_2^4 \leq \frac{3}{4} \zeta_1^{\frac{4}{3}} z_1^4 + \frac{1}{4 \zeta_1^4} \|\tilde{\mathbf{x}}\|^4 \quad (28)$$

where  $\xi_1 > 0$  and  $\zeta_1 > 0$ .

Substituting (22), (27) and (28) into (26) gives

$$\begin{aligned} \mathcal{L}V_1 \leq & -\Xi \|\tilde{\mathbf{x}}\|^4 + \sum_{j=2}^n \varpi_j y^4 + z_1^3 (\hat{x}_1 + \tilde{f}_1) \\ & -\nu(y^2) z_1^4 - \frac{6}{4} z_1^4 + \frac{3}{4} \xi_1^2 - \tilde{\boldsymbol{\theta}}_1^T \Gamma_1^{-1} \dot{\hat{\boldsymbol{\theta}}}_1 \end{aligned} \quad (29)$$

where  $\Xi = \Pi - \frac{1}{4\xi_1^4}$  and  $\tilde{f}_1 = f_1 + k_1 y + \frac{3}{4} \xi_1^3 z_1 + \frac{3}{4\xi_1^2} z_1 \|\mathbf{g}_1\|^4 + \frac{6}{4} z_1 + \nu(y^2) z_1 - \dot{y}_d$ .

**Remark 4.1:**  $\nu(\cdot)$  is a smooth nonnegative function to be designed later. In addition, because the function  $\nu(\cdot)$  is not used for controller design, its true value is not necessary to be known.

Furthermore, by virtue of Lemma 2, an RBFNN can be employed to estimate function  $\tilde{f}_1$ , that is to say, for any given  $\varepsilon_1 > 0$ , there exists a RBFNN  $\boldsymbol{\theta}_1^T S_1(z_1)$ , such that

$$\tilde{f}_1 = \boldsymbol{\theta}_1^T S_1(z_1) + \sigma_1(z_1), |\sigma_1(z_1)| \leq \varepsilon_1 \quad (30)$$

where  $\sigma_1(z_1)$  is the approximation error and  $\mathbf{z}_1 = [z_1]^T$ .

Taking the intermediate control signal  $\alpha_1$  as

$$\alpha_1 = -r_1 z_1^3 - \hat{\boldsymbol{\theta}}_1^T S_1(z_1) \quad (31)$$

where  $r_1 > 0$  is a design parameter.

Considering that  $z_2 = \hat{x}_1 - \alpha_1$ , and according to Young's inequality, we have

$$z_1^3 \tilde{f}_1 \leq z_1^3 \boldsymbol{\theta}_1^T S_1 + \frac{3}{4} z_1^4 + \frac{1}{4} \varepsilon_1^4 \quad (32)$$

$$z_1^3 \hat{x}_1 = z_1^3 (z_2 + \alpha_1) \leq \frac{3}{4} z_1^4 + \frac{1}{4} z_2^4 - r_1 z_1^4 - z_1^3 \hat{\boldsymbol{\theta}}_1^T S_1 \quad (33)$$

Substituting (32) and (33) into (29) gives

$$\begin{aligned} \mathcal{L}V_1 \leq & -\Xi \|\tilde{\mathbf{x}}\|^4 - r_1 z_1^4 + \frac{1}{4} z_2^4 + \sum_{j=2}^n \varpi_j y^4 - \nu(y^2) z_1^4 \\ & + \frac{1}{4} \varepsilon_1^4 + \frac{3}{4} \xi_1^2 + \tilde{\boldsymbol{\theta}}_1^T \left( z_1^3 S_1 - \Gamma_1^{-1} \dot{\hat{\boldsymbol{\theta}}}_1 \right) \end{aligned} \quad (34)$$

*Step  $i$  ( $2 \leq i \leq n-1$ ):* The design step  $i$  ( $2 \leq i \leq n-1$ ) will be elaborated in *Appendix*

A.

*Step n* : Consider a stochastic Lyapunov function candidate as

$$V_n = V_{n-1} + \frac{1}{4} z_n^4 + \frac{1}{2} \tilde{\boldsymbol{\theta}}_n^T \Gamma_n^{-1} \tilde{\boldsymbol{\theta}}_n \quad (35)$$

where  $\tilde{\boldsymbol{\theta}}_n = \boldsymbol{\theta}_n - \hat{\boldsymbol{\theta}}_n$  is the parameter error and  $\Gamma_n = \Gamma_n^T > 0$  is any constant matrix.

By Definition 3, we have

$$\mathcal{L}V_n = \mathcal{L}V_{n-1} + z_n^3 (\hat{x}_n + k_n y - k_{n-1} (\hat{x}_1 + k_1 y) - L_{\alpha_{n-1}}) + \frac{3}{2} z_n^2 \mathbf{g}_n^T \mathbf{g}_n - \tilde{\boldsymbol{\theta}}_n^T \Gamma_n^{-1} \dot{\tilde{\boldsymbol{\theta}}}_n \quad (36)$$

Furthermore, according to *Young's inequality*, we have

$$\frac{3}{2} z_n^2 \mathbf{g}_n^T \mathbf{g}_n \leq \frac{3}{4} \xi_n^2 + \frac{3}{4 \xi_n^2} z_n^4 \|\mathbf{g}_n\|^4 \quad (37)$$

where  $\xi_n > 0$ .

$$\begin{aligned} \mathcal{L}V_n &= \mathcal{L}V_{n-1} + z_n^3 (\hat{x}_n + k_n y - k_{n-1} (\hat{x}_1 + k_1 y) - L_{\alpha_{n-1}}) + \frac{3}{2} z_n^2 \mathbf{g}_n^T \mathbf{g}_n - \tilde{\boldsymbol{\theta}}_n^T \Gamma_n^{-1} \dot{\tilde{\boldsymbol{\theta}}}_n \\ &= \mathcal{L}V_{n-1} + z_n^3 (\hat{x}_n + \tilde{f}_n) - \frac{3}{4} z_n^4 + \frac{3}{4} \xi_n^2 - \tilde{\boldsymbol{\theta}}_n^T \Gamma_n^{-1} \dot{\tilde{\boldsymbol{\theta}}}_n \end{aligned} \quad (38)$$

where  $\tilde{f}_n = k_n y - k_{n-1} (\hat{x}_1 + k_1 y) + \frac{3}{4 \xi_n^2} z_n \|\mathbf{g}_n\|^4 + \frac{3}{4} z_n - L_{\alpha_{n-1}}$  and

$$L_{\alpha_{n-1}} = \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} (x_{j+1} + f_j) + \sum_{j=0}^{n-1} \frac{\partial \alpha_{n-1}}{\partial y_d^{(j)}} y_d^{(j+1)} + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{\boldsymbol{\theta}}_j} \dot{\hat{\boldsymbol{\theta}}}_j + \frac{1}{2} \sum_{p,q=1}^{n-1} \frac{\partial^2 \alpha_{n-1}}{\partial x_p \partial x_q} h_p^T h_q^T.$$

Similarly, for any given  $\varepsilon_n > 0$ , there exists a RBFNN  $\boldsymbol{\theta}_n^T S_n(\mathbf{z}_n)$  such that

$$\tilde{f}_n = \boldsymbol{\theta}_n^T S_n(\mathbf{z}_n) + \sigma_n(\mathbf{z}_n), |\sigma_n(\mathbf{z}_n)| \leq \varepsilon_n \quad (39)$$

where  $\sigma_n(\mathbf{z}_n)$  is the approximation error and  $\mathbf{z}_n = [z_1, \dots, z_n]^T$ .

On the other hand, taking the control signal  $u$  as

$$u = -r_n z_n^3 - \hat{\boldsymbol{\theta}}_n^T S_n(\mathbf{z}_n) \quad (40)$$

where  $r_n > 0$  is a design parameter.

Considering that  $z_{n+1} = \hat{x}_n - u$

$$z_n^3 \tilde{f}_n \leq z_n^3 \tilde{\theta}_n^T S_n + \frac{3}{4} z_n^4 + \frac{1}{4} \varepsilon_n^4 \quad (41)$$

$$z_n^3 \hat{x}_n = z_i^3 u \leq -r_n z_n^4 - z_n^3 \hat{\theta}_n^T S_n \quad (42)$$

Substituting (41) and (42) into (38) gives

$$\begin{aligned} \mathcal{L}V_n &\leq \mathcal{L}V_{n-1} + z_n^3 (\hat{x}_n + \tilde{f}_n) - \frac{3}{4} z_n^4 + \frac{3}{4} \xi_n^2 - \tilde{\theta}_n^T \Gamma_n^{-1} \dot{\hat{\theta}}_n \\ &\leq -\Xi \|\tilde{x}\|^4 - \sum_{j=1}^n r_j z_j^4 + \frac{1}{4} \sum_{j=1}^{n-1} z_{j+1}^4 + \sum_{j=2}^n \varpi_j y^4 - \nu (y^2) z_1^4 \\ &\quad + \frac{1}{4} \sum_{j=1}^n \varepsilon_j^4 + \frac{3}{4} \sum_{j=1}^n \xi_j^2 + \sum_{j=1}^n \tilde{\theta}_j^T \left( z_j^3 S_j - \Gamma_j^{-1} \dot{\hat{\theta}}_j \right) \end{aligned} \quad (43)$$

**Remark 4.2:** Comparing with the results in (Liu, Zhang, & Jiang, 2007), the controller designed in this paper is characterized by simple structure. In addition, in the work of (Wang, Chen, & Lin, 2014; Wang et al., 2015; Wang et al., 2014; Zhou, Shi, Xu, & Li, 2013), only one adaptive parameter is required to be estimated. However, this simplification is only relative to the controller design process and may also lose some approximation precision. Unlike the previous results, in this paper, only a RBFNN is used to estimate unknown nonlinearities in each step of the backstepping. In spite of a large number of neural networks is used in this paper, it will also increase the approximation precision.

## 4.2 Stability analysis

The main results of this paper can be summarized by the following theorem.

**Theorem 4.1:** Consider the stochastic nonlinear system (1) with the observer (13), the intermediate control signals  $\alpha_i (i=1, \dots, n-1)$  described as

$$\alpha_i = -r_i z_i^3 - \hat{\theta}_i^T S_i(z_i) \quad (44)$$

the control law described as (40) and the adaptive laws defined as

$$\dot{\hat{\theta}}_i = -\Gamma_i \eta_i \hat{\theta}_i + \Gamma_i z_i^3 S_i \quad (45)$$

where  $r_i > 0$  and  $\eta_i > 0$  are designed parameters and matrices  $\Gamma_i = \Gamma_i^T > 0$ . Then, for bounded initial conditions, all the signals in the closed-loop system remain bounded in probability and the tracking error finally converges to a small domain around the origin in the sense of mean quartic value.

*Proof:* Consider the stochastic Lyapunov function as

$$V = \frac{1}{2} (\tilde{\mathbf{x}}^T P \tilde{\mathbf{x}})^2 + \frac{1}{4} \sum_{j=1}^n z_j^4 + \frac{1}{2} \sum_{j=1}^n \tilde{\theta}_j^T \Gamma_j^{-1} \tilde{\theta}_j \quad (46)$$

It follows from (43) that

$$\begin{aligned} \mathcal{L}V &\leq -\Xi \|\tilde{\mathbf{x}}\|^4 - \sum_{j=1}^n r_j z_j^4 + \frac{1}{4} \sum_{j=1}^{n-1} z_{j+1}^4 + \sum_{j=2}^n \varpi_j y^4 - \nu(y^2) z_1^4 \\ &\quad + \frac{1}{4} \sum_{j=1}^n \varepsilon_j^4 + \frac{3}{4} \sum_{j=1}^n \xi_j^2 + \sum_{j=1}^n \tilde{\theta}_j^T (z_j^3 S_j - \Gamma_j^{-1} \dot{\hat{\theta}}_j) \end{aligned} \quad (47)$$

For the term  $\sum_{j=1}^n \tilde{\theta}_j^T (z_j^3 S_j - \Gamma_j^{-1} \dot{\hat{\theta}}_j)$ , we have

$$\begin{aligned} \sum_{j=1}^n \tilde{\theta}_j^T (z_j^3 S_j - \Gamma_j^{-1} \dot{\hat{\theta}}_j) &= \sum_{j=1}^n \eta_j \tilde{\theta}_j^T \hat{\theta}_j \\ &\leq -\sum_{j=1}^n \frac{\eta_j}{2} \tilde{\theta}_j^T \tilde{\theta}_j + \sum_{j=1}^n \frac{\eta_j}{2} \|\hat{\theta}_j\|^2 \\ &\leq -\sum_{j=1}^n \frac{\eta_j}{2\lambda_{\max}(\Gamma_j^{-1})} \tilde{\theta}_j^T \Gamma_j^{-1} \tilde{\theta}_j + \sum_{j=1}^n \frac{\eta_j}{2} \|\hat{\theta}_j\|^2 \end{aligned} \quad (48)$$

Choose a smooth nonnegative function candidate  $\nu(y^2)$  such that

$$\sum_{j=2}^n \varpi_j y^4 - \nu(y^2) z_1^4 \leq 0 \quad (49)$$

Subsequently, combining (47) with (48) and (49) results in

$$\begin{aligned} \mathcal{L}V \leq & -\Xi \|\tilde{\mathbf{x}}\|^4 - \sum_{j=1}^n \left( r_j - \frac{1}{4} \right) z_j^4 - \sum_{j=1}^n \frac{\eta_j}{2\lambda_{\max}(\Gamma_j^{-1})} \tilde{\boldsymbol{\theta}}_j^T \Gamma_j^{-1} \tilde{\boldsymbol{\theta}}_j \\ & + \frac{1}{4} \sum_{j=1}^n \varepsilon_j^4 + \frac{3}{4} \sum_{j=1}^n \xi_j^2 + \sum_{j=1}^n \frac{\eta_j}{2} \|\hat{\boldsymbol{\theta}}_j\|^2 \end{aligned} \quad (50)$$

Let

$$\begin{aligned} a_0 &= \min_{1 \leq j \leq n} \left\{ \frac{2\Xi}{\lambda_{\max}^2(P)}, 4 \left( r_j - \frac{1}{4} \right), \frac{\eta_j}{\lambda_{\max}(\Gamma_j^{-1})} \right\} \\ b_0 &= \frac{1}{4} \sum_{j=1}^n \varepsilon_j^4 + \frac{3}{4} \sum_{j=1}^n \xi_j^2 + \sum_{j=1}^n \frac{\eta_j}{2} \|\hat{\boldsymbol{\theta}}_j\|^2 \end{aligned}$$

yields

$$\mathcal{L}V \leq -a_0 V + b_0 \quad (51)$$

Based on Lemma 1, inequality (51) further implies that

$$\frac{dE[V(t)]}{dt} \leq -a_0 E[V(t)] + b_0 \quad (52)$$

Based on (52) and Lemma 1, it follows that all the signals of the closed-loop system bounded in probability. Moreover, by adjusting the design parameter, the tracking error can be made arbitrarily small in the sense of mean quartic value.

**Remark 4.3:** In theory, according to Theorem 4.1, for stochastic nonlinear system (1), a good tracking control performance can be obtained by choosing appropriate design parameter for (31), (40), (44) and (45). However, in practical applications, in order to achieve a desired tracking performance and transient performance, the above parameters should be adjusted carefully.

## 5 Simulation examples

In this section, two numerical simulation examples are given to illustrate the effectiveness of the proposed control approach.

*Example 1.* Consider the following third-order stochastic nonlinear system

$$\begin{cases} dx_1 = (x_2 - y + y^2)dt + 0.1 \sin y d\omega \\ dx_2 = (x_3 + y^2 \cos y)dt + \frac{y}{1+y^2} d\omega \\ dx_3 = (u + \sin y e^{-y^2})dt + 0.1 \cos y d\omega \\ y = x_1 \end{cases} \quad (53)$$

where  $x_1$ ,  $x_2$  and  $x_3$  are the state variables,  $u$  is the control input and  $y$  is the system output. The control objective is to design an adaptive neural controller such that the system output  $y$  follows the given reference signal  $y_d = 0.5(\sin(t) + \sin(0.5t))$ .

According to Theorem 4.1, the intermediate control signals  $\alpha_1$ ,  $\alpha_2$  and the control law  $u$  are designed as

$$\alpha_i = -r_i z_i^3 - \hat{\theta}_i^T S_i(z_i) \quad i=1,2 \quad (54)$$

$$u = -r_3 z_3^3 - \hat{\theta}_3^T S_3(z_3) \quad (55)$$

where  $z_1 = x_1 - y_d$ ,  $z_2 = x_1 - \alpha_1$ ,  $z_3 = x_2 - \alpha_2$  and  $z_1 = [z_1]$ ,  $z_2 = [z_1, z_2]^T$ ,  $z_3 = [z_1, z_2, z_3]^T$ . The adaptive laws are designed as

$$\dot{\hat{\theta}}_i = -\Gamma_i \eta_i \hat{\theta}_i + \Gamma_i z_i^3 S_i, \quad i=1,2,3 \quad (56)$$

In the simulation,  $\hat{\theta}_1^T S_1(z_1)$  contains 7 nodes with centres spaced evenly in the interval  $[-3, 3]$  and widths being equal to 10;  $\hat{\theta}_2^T S_2(z_2)$  contains 9 nodes with centres spaced evenly in the interval  $[-8, 8] \times [-8, 8]$  and widths being equal to 10;  $\hat{\theta}_3^T S_3(z_3)$  contains 9 nodes with centres spaced evenly in the interval  $[-8, 8] \times [-8, 8] \times [-8, 8]$  and widths being equal to 10. The design parameters are chosen as:  $r_1 = 15$ ,  $r_2 = 10$ ,  $r_3 = 20$ ,  $\eta_1 = 1.5$ ,  $\eta_2 = 2$ ,  $\eta_3 = 5$  and  $\Gamma_1 = 30I_7$ ,  $\Gamma_2 = I_9$ ,  $\Gamma_3 = 40I_9$ .

The simulation is run with the initial conditions  $[x_1(0), x_2(0), x_3(0)]^T = [0, 0, 0]^T$  and the simulation results are illustrated in Figures 1-5, respectively. Figure 1 shows the system output  $y$  and the reference signal  $y_d$ , it can be seen that the good tracking performance has been achieved. Figure 2 displays the control input signal  $u$ . Figure 3 shows that the state variables  $x_1$ ,  $x_2$  and  $x_3$  are bounded. Figure 4 shows that the observer state variables  $\hat{x}_1$  and  $\hat{x}_2$  are bounded. Figure 5 depicts that the tracking error finally converges to a small domain around the origin.

*(The position of Figures 1-5)*

From Figures 2-5, it is obvious that all the signals of the closed-loop system are bounded in probability. The simulation results illustrate the effectiveness of the observer-NN-based adaptive tracking control approach proposed in this paper.

**Remark 5.1:** It is important to note that the tracking error can be made arbitrarily small by choosing appropriately the design parameters  $r_i$ ,  $\eta_i$  and  $\Gamma_i$ . In addition, simulation results shown that the design parameters  $r_i$  influence the tracking performance for the closed-loop system, therefore, we should appropriately select the value of the parameters  $r_i$  in practical application.

*Example 2.* Consider the following third-order stochastic nonlinear system

$$\begin{cases} dx_1 = (x_2 + y^2)dt + 0.5 \sin y \cos y d\omega \\ dx_2 = (x_3 - y^3 + \sin y)dt + \frac{y}{1+y^2} d\omega \\ dx_3 = (u + y^2 e^{-y^2})dt + 0.5 \sin y \cos y d\omega \\ y = x_1 \end{cases} \quad (57)$$

where  $x_1$ ,  $x_2$  and  $x_3$  are the state variables,  $u$  is the control input and  $y$  is the system output. The control objective is to design an adaptive neural controller such that the system output  $y$  follows the given reference signal  $y_d = 0.5(\sin(0.5t) + \sin(0.25t))$ .

Similarly, Theorem 4.1 is used to design adaptive neural controller for this stochastic nonlinear system (57). Now, we further apply (54), (55) and (56) to system (57), and all the control parameters are kept as that in Example 1. The simulation is run with the initial conditions  $[x_1(0), x_2(0), x_3(0)]^T = [0, 0, 0]^T$  and the simulation results are illustrated in Figures 6-10, respectively. It can be seen that the control performance is still fairly satisfactory, which further verify illustrate the effectiveness of the adaptive output feedback control approach proposed in this paper.

*(The position of Figures 6-10)*

**Remark 5.2:** By the Examples 1 and 2, one can conclude that the proposed observer-based adaptive neural tracking control approach can achieve the good control performance for different third-order stochastic nonlinear system under the same control parameters. Thus, we can conclude that the control method proposed in this paper has a better tracking control performance and generalized ability.

**Remark 5.3:** Compared with the results in (Wang et al., 2015), Examples 1 and 2 indicate that the main advantages of this paper are that the proposed method requires fewer nodes and can get precise tracking results with low computational cost.

## 6 Conclusions

In this paper, an adaptive neural output feedback control approach is developed for a class of stochastic nonlinear systems with immeasurable states. NNs are utilized to approximate the unknown nonlinear functions, a linear reduced-order observer is introduced to estimate the immeasurable states, and then a new adaptive neural output feedback control scheme is developed via backstepping technique. The proposed control approach can guarantee that all the signals of the closed-loop system are bounded in probability, and also that the tracking errors converge to a small neighborhood of the origin in the sense of mean quartic value. Finally, two numerical examples are used to demonstrate the effectiveness of the proposed approach.

Our future research will be directed at further extending the proposed methodology to a broader class of stochastic nonlinear systems. For example, the functions  $f_i(\cdot)$  in system (1) rely on state  $x_1$ , which limits the range of application. Therefore, it is a meaningful work to extend proposed approach to general stochastic nonlinear system.

Other issues under current investigation include the robust problem and adaptive output-feedback control for large-scale stochastic nonlinear systems.

## Appendix A

The  $i(2, \dots, n-1)$  step of Backstepping in Section 4.1

Step 2: Consider a stochastic Lyapunov function candidate as

$$V_2 = V_1 + \frac{1}{4}z_2^4 + \frac{1}{2}\tilde{\theta}_2^T \Gamma_2^{-1} \tilde{\theta}_2 \quad (\text{A.1})$$

where  $\tilde{\theta}_2 = \theta_2 - \hat{\theta}_2$  is the parameter error and  $\Gamma_2 = \Gamma_2^T > 0$  is any constant matrix.

By Definition 3, we have

$$\mathcal{L}V_2 = \mathcal{L}V_1 + z_2^3 (\hat{x}_2 + k_2 y - k_1 (\hat{x}_1 + k_1 y) - L_{\alpha_1}) + \frac{3}{2}z_2^2 g_2^T g_2 - \tilde{\theta}_2^T \Gamma_2^{-1} \dot{\tilde{\theta}}_2 \quad (\text{A.2})$$

Furthermore, according to *Young's inequality*, we have

$$\frac{3}{2}z_2^2 g_2^T g_2 \leq \frac{3}{4}\xi_2^2 + \frac{3}{4\xi_2^2}z_2^4 \|g_2\|^4 \quad (\text{A.3})$$

where  $\xi_{1,2} > 0$ .

$$\begin{aligned} \mathcal{L}V_2 &\leq \mathcal{L}V_1 + z_2^3 \left( \hat{x}_2 + k_2 y - k_1 (\hat{x}_1 + k_1 y) + \frac{3}{4\xi_2^2}z_2^4 \|g_2\|^4 - L_{\alpha_1} \right) \\ &\quad + \frac{3}{4}\xi_{1,2}^2 - \tilde{\theta}_2^T \Gamma_2^{-1} \dot{\tilde{\theta}}_2 \\ &= \mathcal{L}V_1 + z_2^3 (\hat{x}_2 + \tilde{f}_2) - \frac{6}{4}z_2^4 + \frac{3}{4}\xi_2^2 - \tilde{\theta}_2^T \Gamma_2^{-1} \dot{\tilde{\theta}}_2 \end{aligned} \quad (\text{A.4})$$

where  $\tilde{f}_2 = k_2 y - k_1 (\hat{x}_1 + k_1 y) + \frac{3}{4\xi_2^2}z_2^4 \|g_2\|^4 + \frac{6}{4}z_2^4 - L_{\alpha_1}$  and

$$L_{\alpha_1} = \frac{\partial \alpha_1}{\partial x_1} (x_2 + f_1) + \sum_{j=0}^1 \frac{\partial \alpha_{i-1}}{\partial y_d^{(j)}} y_d^{(j+1)} + \frac{\partial \alpha_1}{\partial \hat{\theta}_1} \dot{\hat{\theta}}_1 + \frac{1}{2} \frac{\partial^2 \alpha_1}{\partial x_1 \partial x_1} h_1^T h_1.$$

Similarly, for any given  $\varepsilon_2 > 0$ , there exists a RBFNN  $\theta_2^T S_2(z_2)$  such that

$$\tilde{f}_2 = \theta_2^T S_2(z_2) + \sigma_2(z_2), |\sigma_2(z_2)| \leq \varepsilon_2 \quad (\text{A.5})$$

where  $\sigma_2(z_2)$  is the approximation error and  $z_2 = [z_1, z_2]^T$ .

On the other hand, taking the intermediate control signal  $\alpha_2$  as

$$\alpha_2 = -r_2 z_2^3 - \hat{\theta}_2^T S_2(z_2) \quad (\text{A.6})$$

where  $r_2 > 0$  is a design parameter.

Considering that  $z_3 = \hat{x}_2 - \alpha_2$  and according to *Young's inequality*, we have

$$z_2^3 \tilde{f}_2 \leq z_2^3 \boldsymbol{\theta}_2^T S_2 + \frac{3}{4} z_2^4 + \frac{1}{4} \varepsilon_2^4 \quad (\text{A.7})$$

$$z_2^3 \hat{x}_2 = z_2^3 (z_3 + \alpha_2) \leq \frac{3}{4} z_2^4 + \frac{1}{4} z_3^4 - r_2 z_2^4 - z_2^3 \hat{\boldsymbol{\theta}}_2^T S_2 \quad (\text{A.8})$$

Substituting (34), (A.7) and (A.8) into (A.4) gives

$$\begin{aligned} \mathcal{L}V_2 \leq & -\Xi \|\tilde{\mathbf{x}}\|^4 - \sum_{j=1}^2 r_j z_j^4 + \frac{1}{4} \sum_{j=1}^2 z_{j+1}^4 + \sum_{j=2}^n \omega_j y^4 - \nu (y^2) z_1^4 \\ & + \frac{1}{4} \sum_{j=1}^2 \varepsilon_j^4 + \frac{3}{4} \sum_{j=1}^2 \xi_j^2 + \sum_{j=1}^2 \tilde{\boldsymbol{\theta}}_j^T \left( z_j^3 S_j - \Gamma_j^{-1} \dot{\hat{\boldsymbol{\theta}}}_j \right) \end{aligned} \quad (\text{A.9})$$

*Step i* ( $2 \leq i \leq n-1$ ): Consider a stochastic Lyapunov function candidate as

$$V_i = V_{i-1} + \frac{1}{4} z_i^4 + \frac{1}{2} \tilde{\boldsymbol{\theta}}_i^T \Gamma_i^{-1} \tilde{\boldsymbol{\theta}}_i \quad (\text{A.10})$$

where  $\tilde{\boldsymbol{\theta}}_i = \boldsymbol{\theta}_i - \hat{\boldsymbol{\theta}}_i$  is the parameter error and  $\Gamma_i = \Gamma_i^T > \mathbf{0}$  is any constant matrix.

By Definition 3, we have

$$\mathcal{L}V_i = \mathcal{L}V_{i-1} + z_i^3 \left( \hat{x}_i + k_i y - k_{i-1} (\hat{x}_1 + k_1 y) - L_{\alpha_{i-1}} \right) + \frac{3}{2} z_i^2 \mathbf{g}_i^T \mathbf{g}_i - \tilde{\boldsymbol{\theta}}_i^T \Gamma_i^{-1} \dot{\hat{\boldsymbol{\theta}}}_i \quad (\text{A.11})$$

Furthermore, according to *Young's inequality*, we have

$$\frac{3}{2} z_i^2 \mathbf{g}_i^T \mathbf{g}_i \leq \frac{3}{4} \xi_i^2 + \frac{3}{4 \xi_i^2} z_i^4 \|\mathbf{g}_i\|^4 \quad (\text{A.12})$$

where  $\xi_i > 0$ .

$$\begin{aligned} \mathcal{L}V_i &= \mathcal{L}V_{i-1} + z_i^3 \left( \hat{x}_i + k_i y - k_{i-1} (\hat{x}_1 + k_1 y) - L_{\alpha_{i-1}} \right) + \frac{3}{2} z_i^2 \mathbf{g}_i^T \mathbf{g}_i - \tilde{\boldsymbol{\theta}}_i^T \Gamma_i^{-1} \dot{\hat{\boldsymbol{\theta}}}_i \\ &= \mathcal{L}V_{i-1} + z_i^3 \left( \hat{x}_i + \tilde{f}_i \right) - \frac{6}{4} z_i^4 + \frac{3}{4} \xi_i^2 - \tilde{\boldsymbol{\theta}}_i^T \Gamma_i^{-1} \dot{\hat{\boldsymbol{\theta}}}_i \end{aligned} \quad (\text{A.13})$$

where  $\tilde{f}_i = k_i y - k_{i-1} (\hat{x}_1 + k_1 y) + \frac{3}{4 \xi_i^2} z_i \|\mathbf{g}_i\|^4 + \frac{6}{4} z_i - L_{\alpha_{i-1}}$  and

$$L_{\alpha_{i-1}} = \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} (x_{j+1} + f_j) + \sum_{j=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(j)}} y_d^{(j+1)} + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{\boldsymbol{\theta}}_j} \dot{\hat{\boldsymbol{\theta}}}_j + \frac{1}{2} \sum_{p,q=1}^{i-1} \frac{\partial^2 \alpha_{i-1}}{\partial x_p \partial x_q} h_p^T h_q.$$

Similarly, for any given  $\varepsilon_i > 0$ , there exists a RBFNN  $\boldsymbol{\theta}_i^T S_i(\mathbf{z}_i)$  such that

$$\tilde{f}_i = \boldsymbol{\theta}_i^T S_i(\mathbf{z}_i) + \sigma_i(\mathbf{z}_i), \quad |\sigma_i(\mathbf{z}_i)| \leq \varepsilon_i \quad (\text{A.14})$$

where  $\sigma_i(\mathbf{z}_i)$  is the approximation error and  $\mathbf{z}_i = [z_1, \dots, z_i]^T$ .

On the other hand, taking the intermediate control signal  $\alpha_i$  as

$$\alpha_i = -r_i z_i^3 - \hat{\theta}_i^T S_i(z_i) \quad (\text{A.15})$$

where  $r_i > 0$  is a design parameter.

Considering that  $z_{i+1} = \hat{x}_i - \alpha_i$  and according to *Young's inequality*, we have

$$z_i^3 \tilde{f}_i \leq z_i^3 \theta_i^T S_i + \frac{3}{4} z_i^4 + \frac{1}{4} \varepsilon_i^4 \quad (\text{A.16})$$

$$z_i^3 \hat{x}_i = z_i^3 (z_{i+1} + \alpha_i) \leq \frac{3}{4} z_i^4 + \frac{1}{4} z_{i+1}^4 - r_i z_i^4 - z_i^3 \hat{\theta}_i^T S_i \quad (\text{A.17})$$

Substituting (A.16) and (A.17) into (A.13) gives

$$\begin{aligned} \mathcal{L}V_{i-1} \leq & -\Xi \|\tilde{x}\|^4 - \sum_{j=1}^i r_j z_j^4 + \frac{1}{4} \sum_{j=1}^i z_{j+1}^4 + \sum_{j=2}^n \varpi_j y^4 - \nu(y^2) z_1^4 \\ & + \frac{1}{4} \sum_{j=1}^i \varepsilon_j^4 + \frac{3}{4} \sum_{j=1}^i \xi_j^2 + \sum_{j=1}^i \tilde{\theta}_j^T \left( z_j^3 S_j - \Gamma_j^{-1} \dot{\hat{\theta}}_j \right) \end{aligned} \quad (\text{A.18})$$

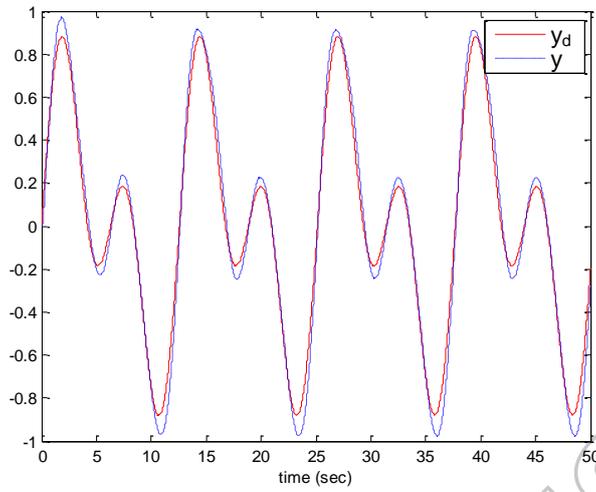
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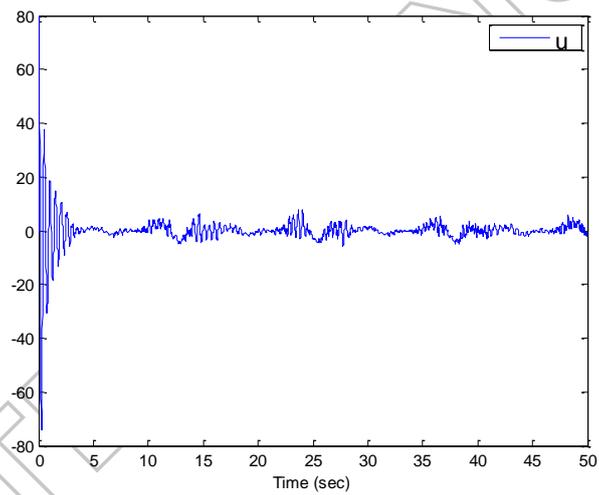
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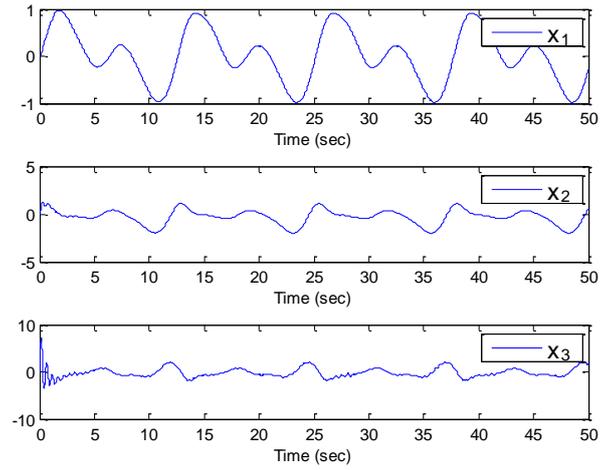
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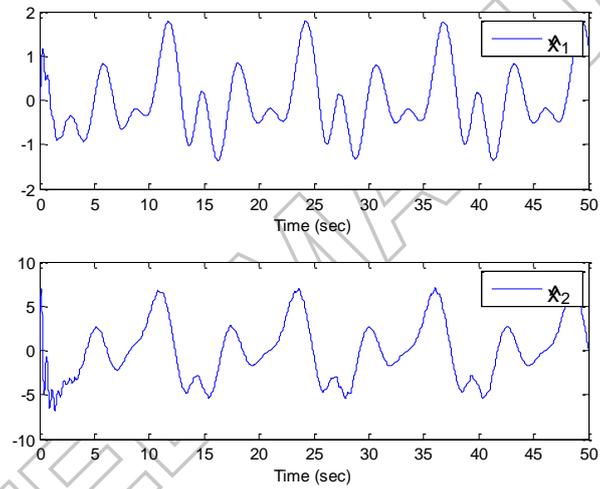
**Figure 1.** System output  $y$  and reference signal  $y_d$  of Example 1.



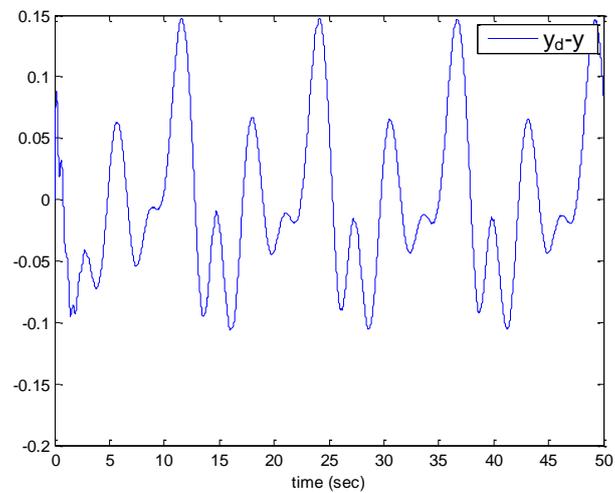
**Figure 2.** Control input  $u$  of Example 1.



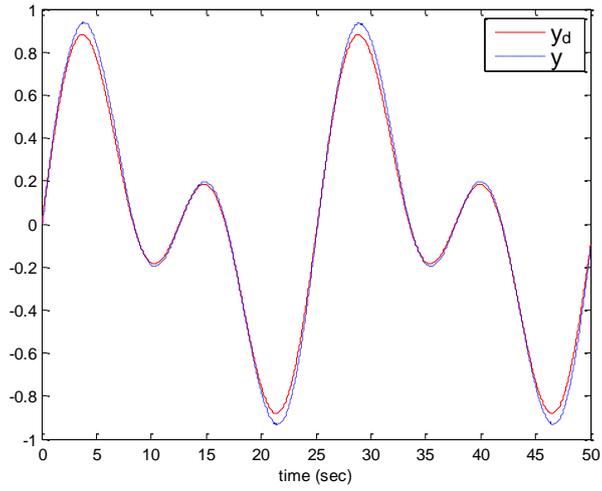
**Figure 3.** System state variables  $x_1$ ,  $x_2$  and  $x_3$  of Example 1.



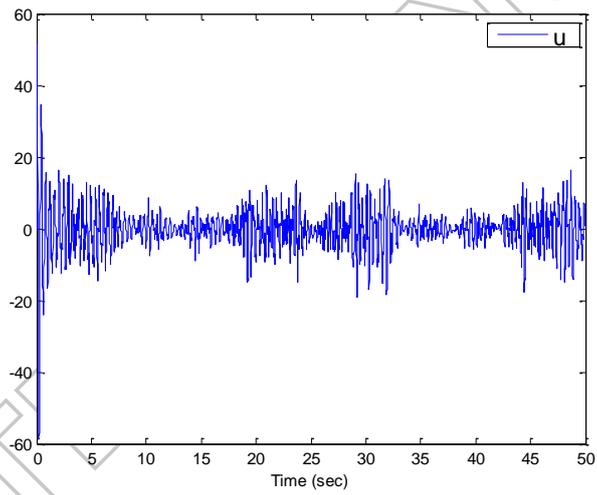
**Figure 4.** Observer state variables  $\hat{x}_1$  and  $\hat{x}_2$  of Example 1.



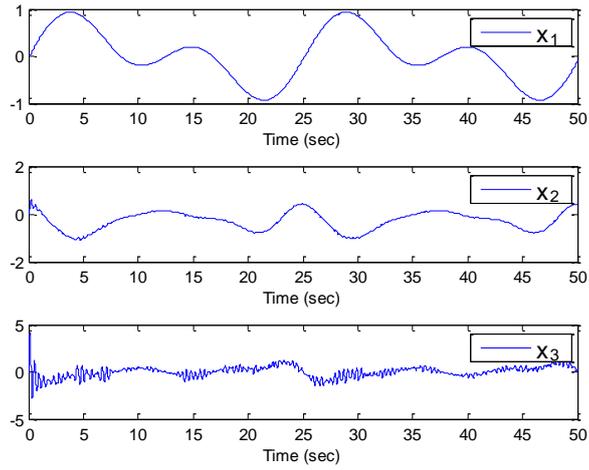
**Figure 5.** Tracking error  $y_d - y$  of Example 1.



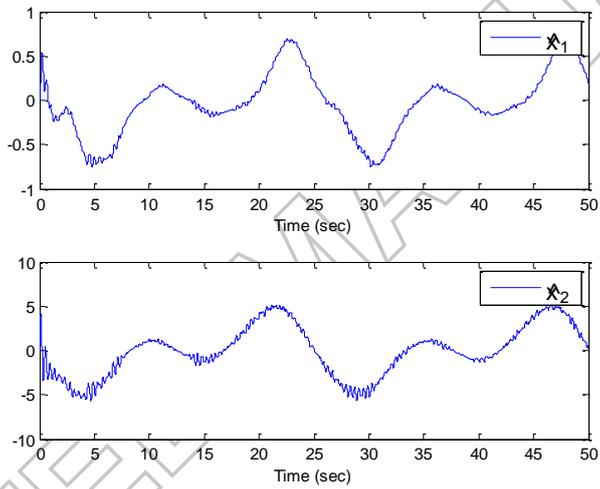
**Figure 6.** System output  $y$  and reference signal  $y_d$  of Example 2.



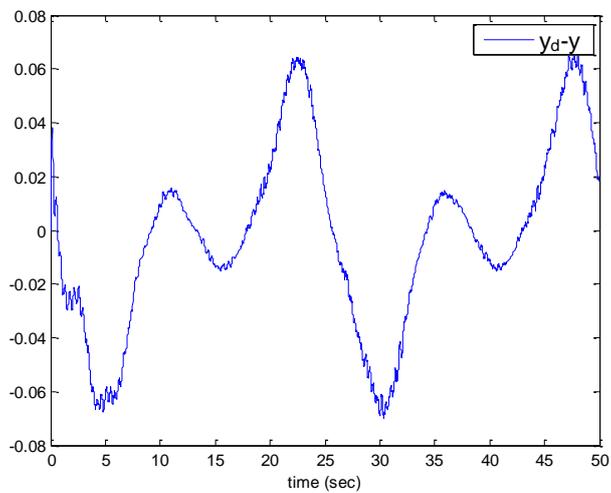
**Figure 7.** Control input  $u$  of Example 2.



**Figure 8.** State variables  $x_1$ ,  $x_2$  and  $x_3$  of Example 2.



**Figure 9.** State variables  $\hat{x}_1$  and  $\hat{x}_2$  of Example 2.



**Figure 10.** Tracking error  $y_d - y$  of Example 2.